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Flow due to an oscillatory free stream past a porous rotating disk

R. PURUSHOTHAMAN AND P. RAJAGOPALAN

Department of Mathematics, Regional Engineering College, Tiruchirapalli-620015, India

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SUMMARY

The effects of constant suction or injection of a homogeneous viscous fluid on the oscillatory flow past a porous disk are studied. Due to the oscillatory free stream, a secondary flow with the tangential components of velocity alone is produced. Separate solutions for low and high frequencies of oscillation are obtained. The streamwise and crosswise components of the oscillatory flow are obtained by choosing suitable forms for the low- and high-frequency oscillations. Expressions for the correction to the shearing stress at the surface due to the secondary flow are presented.

1. Introduction

Viscous fluid flow due to a rotating disk has been studied extensively by Von Kármán [1], Cochran [2], Fettis [3], Benton [4], Zandbergen and Dijkstra [5]. The effects of suction and injection of the fluid through the rotating disk were investigated by Stuart [6], Sparrow and Gregg [7], Rogers and Lance [8], Kuiken [9] and Chawla [10]. In all these studies, Von Kármán's similarity transformation is used to reduce the Navier-Stokes equations for steady axisymmetric flow to a set of ordinary differential equations, which is solved using analytical and numerical methods. The axial symmetry of the flow gets destroyed if translational velocities are superposed on the axisymmetric flow. Rott and Lewellen [11] considered a class of flows in which the axisymmetric nature of the flow is changed by the interaction of translational velocities. Purushothaman [12] extended the analysis of Rott and Lewellen [11] by considering oscillations in the free stream past a rotating disk. The fluid flow chosen in [12] consists of two types of motion, viz. an axisymmetric steady flow due to the rotating disk and an oscillatory flow due to the free stream. The oscillatory flow does not contain the normal component of the velocity. Benton's [4] solution was used as the basic solution. In the present paper, we study the effects of constant suction and injection of the fluid through a porous rotating disk. With Sparrow and Gregg's [7] solution for the primary steady flow as the starting point, the secondary flow velocities are determined. The governing differential equations for the secondary motion contain a frequency parameter $\beta = \omega/\Omega$, where ω is the frequency of oscillation and Ω is the constant angular velocity of the disk. Choosing appropriate forms for the flow functions, expressions for the streamwise and crosswise components are obtained, for small and large values of the frequency parameter. When $\beta = 0$, we obtain the steady flow due to constant free stream past the porous rotating disk. In view of the presence of the Stokes shear layer in the case of large frequencies of oscillation, the determination of the velocity componetns becomes a singular perturbation problem. The method of composite expansions is employed to investigate the high-frequency flow. The corrections to the skin-friction components at the surface of the disk provide the limitation of the present analysis.

2. Formulation of the problem

We choose a Cartesian co-ordinate system in which the bounding disk is $z = 0$, the axis of rotation is the z-axis and the x-axis is parallel to the direction of the free stream. Let Ω be the constant angular velocity of the disk, w_0 the constant normal velocity of the fluid at the disk and $U_0 \cos \omega t$, the velocity of the oscillating free stream. Following Von Kármán [1] and Lin [13], the velocity components (u, v, w) and the pressure p are chosen in the form

$$u = \Omega\left[-\frac{x}{2}\frac{dH}{d\eta} - yG(\eta)\right] + U_0 H_1(\eta) e^{i\beta\tau}, \tag{2.1}$$

$$v = \Omega\left[xG(\eta) - \frac{y}{2}\frac{dH}{d\eta}\right] + U_0 H_2(\eta) e^{i\beta\tau}, \tag{2.2}$$

$$w = (\nu\Omega)^{1/2} H(\eta), \tag{2.3}$$

$$p = -\rho\nu\left[\Omega P(\eta) + U_0 i\beta x e^{i\beta\tau}\right], \tag{2.4}$$

where $\eta = z(\Omega/\nu)^{1/2}$, $\tau = \Omega t$, $\beta = \omega/\Omega$. The physical quantities ρ and ν are the density and kinematic coefficient of viscosity of the fluid. Wherever complex quantities occur, only the real part has a physical significance. Substituting (2.1) to (2.4) in the Navier-Stokes equations of motion and considering the terms proportional to x, proportional to y and those independent of x and y separately, we obtain the following system of differential equations for the functions H, G, H_1 and G_1:

$$H''' - HH'' + \tfrac{1}{2}(H')^2 - 2G^2 = 0, \tag{2.5}$$

$$G'' - HG' + H'G = 0, \tag{2.6}$$

$$H_1'' - HH_1' + \tfrac{1}{2}H'H_1 + GH_2 - i\beta H_1 = -i\beta, \tag{2.7}$$

$$H_2'' - HH_2' + \tfrac{1}{2}H'H_2 - GH_1 - i\beta H_2 = 0, \tag{2.8}$$

where a prime denotes differentiation with respect to η. The functions H and G correspond to the primary steady flow due to a rotating disk, whereas the functions H_1 and H_2 refer to the secondary flow due to the superposed translational velocity of the fluid at large distances from the disk. The structure of (2.1) to (2.3) shows that the normal velocity is not affected by the secondary flow. The secondary flow contributes only to the radial and circumferential velocities. The appropriate boundary conditions for the problem are

$$H(0) = a, \quad H'(0) = 0, \quad G(0) = 1, \quad H'(\infty) = 0, \quad G(\infty) = 0, \tag{2.9}$$

$$H_1(0) = H_2(0) = 0, \quad H_1(\infty) = 1, \quad H_2(\infty) = 0, \tag{2.10}$$

where $a = w_0/\sqrt{\nu\Omega}$ is a constant. A positive value of a denotes injection and a negative value of a means suction. The solutions of (2.5) and (2.6) with (2.9), for various values of a, have been obtained by Stuart [6] and Sparrow and Gregg [7]. Stuart [6] has obtained a series solution for the functions H and G which holds for $a \geqslant 1$, whereas Sparrow and Gregg [7] have solved the equations numerically for different values of a in the range $-3 \leqslant a \leqslant 3$. Kuiken [9] discussed the effects of strong blowing on the flow due to the rotating disk. From (2.7) and (2.8) we observe that the secondary-flow functions H_1 and H_2 depend on the functions H and G. The solution of Sparrow and Gregg [7] is adapted for the determination of the functions H_1 and H_2. Also the secondary flow depends on the frequency of oscillation of the external stream. The effects of small and large frequencies of oscillation are discussed in the following sections.

3. Solution for small frequencies

When the frequency of oscillations of the free stream is small, ($\beta \ll 1$) we can expand the functions H_1 and H_2 in the form

$$(H_1, H_2) = \sum_{n=0}^{\infty} (i\beta)^n (H_{1n}, H_{2n}). \tag{3.1}$$

Substituting (3.1) in (2.7), (2.8) and (2.10) and considering like powers of β, we obtain the following system of equations:

$$H''_{10} - HH'_{10} + \tfrac{1}{2}H'H_{10} + GH_{20} = 0, \tag{3.2}$$

$$H''_{20} - HH'_{20} + \tfrac{1}{2}H'H_{20} - GH_{10} = 0, \tag{3.3}$$

$$H''_{11} - HH'_{11} + \tfrac{1}{2}H'H_{11} + GH_{21} = H_{10} - 1, \tag{3.4}$$

$$H''_{21} - HH'_{21} + \tfrac{1}{2}H'H_{21} - GH_{11} = H_{21}, \tag{3.5}$$

$$H''_{1n} - HH'_{1n} + \tfrac{1}{2}H'H_{1n} + GH_{2n} = H_{1(n-1)}, \tag{3.6}$$

$$H''_{2n} - HH'_{2n} + \tfrac{1}{2}H'H_{2n} - GH_{1n} = H_{2(n-1)}, \qquad n \geqslant 2, \tag{3.7}$$

Table 1.

a	$H'_{10}(0)$	$H'_{20}(0)$	$H'_{11}(0)$	$H'_{21}(0)$	$H'_{12}(0)$	$H'_{22}(0)$
– 4.0	3.99646	– 0.13832	0.27614	0.00883	– 0.01729	1.15816
– 3.0	2.99233	– 0.17805	0.36010	0.01953	– 0.04017	– 0.00186
– 2.5	2.48762	– 0.20590	0.42799	0.02664	– 0.06874	– 0.00569
– 2.0	1.97850	– 0.24504	0.54121	0.04655	– 0.13278	– 0.00310
– 1.5	1.46188	– 0.29421	0.70176	0.09343	– 0.29801	– 0.00986
– 1.2	1.15022	– 0.31729	0.85622	0.12484	– 0.51317	– 0.06264
– 1.0	0.94740	– 0.33034	0.97799	0.17294	– 0.72242	– 0.11684
– 0.8	0.75346	– 0.31440	1.11257	0.12250	– 1.00348	– 0.28169
– 0.4	0.42516	– 0.25100	1.31938	– 0.02249	– 1.54090	– 0.13637
– 0.2	0.30262	– 0.20837	1.34172	– 0.10650	– 1.61251	0.24735
0.0	0.21083	– 0.16135	1.28889	– 0.20224	– 1.43098	0.45435
0.1	0.15789	– 0.12425	1.34856	– 0.32369	– 1.17221	1.15816
0.3	0.11059	– 0.10326	1.19143	– 0.26969	– 1.26217	0.81886
0.5	0.07062	– 0.07222	1.07667	– 0.28401	– 0.98452	0.78726
0.6	0.05189	– 0.05868	1.03305	– 0.29325	– 0.92767	0.79006
0.7	0.03389	– 0.05038	1.05734	– 0.25655	– 0.53217	0.5323

with

$$H_{10}(0) = 0, \quad H_{20}(0) = 0, \quad H_{10}(\infty) = 1, \quad H_{20}(\infty) = 0, \tag{3.8}$$

$$H_{1n}(0) = H_{2n}(0) = 0, \quad H_{1n}(\infty) = H_{2n}(\infty) = 0, \qquad n \geqslant 1. \tag{3.9}$$

The two-point boundary value problems defined in (3.2) to (3.8) are solved by the method of complementary functions [14] using a Runge-Kutta-Gill routine. The initial values for the primary profiles H and G are taken from Sparrow and Gregg [7]. The missing initial conditions $H'_{1n}(0), H'_{2n}(0), n = 0, 1, 2$, for various values of a are shown in Table 1.

The secondary flow due to a constant free stream can be obtained from the limiting case of $\beta \to 0$. As $\beta \to 0$, $H_1 = H_{10}$, $H_2 = H_{20}$. The quasi-steady flow in the case of $a = 0$ is in good agreement with the result of Rott and Lewellen [11] and Purushothaman [12]. The secondary-flow functions H_{10}, H_{20} for a few representative values of the parameter a are shown in Figures 1 and 2. The functions H_{10} and H_{20} are similar to the primary-flow functions G and H. From the form of H_{10}, we observe that the profile has a point of inflexion when a is large and positive. As in the case of the primary flow, when the blowing is strong, transition to turbulence for small Reynolds numbers can be expected. The thickness of the boundary layer in the secondary flow increases when there is injection ($a > 0$) and decreases when suction ($a < 0$) is applied. The shear associated with the cross component ($H'_{20}(0)$) is in the negative y-direction for all values of a. The magnitude of $H'_{20}(0)$ is maximum when $a = -1.0$ and vanishes when a is large. But the windwise component of the shear (represented by $H'_{10}(0)$) steadily decreases as a increases and tends to zero for large a. When the fluid is blown strongly into the fluid, the secondary flow has no effect on the shearing stresses on the surface. In fact the secondary flow is insignificant in comparison with the primary flow. The cross component of the shear is as big as the windwise component of shear when there is blowing. But the windwise shear is larger than the crosswise shear when the fluid is sucked through the surface. It is interesting to note that

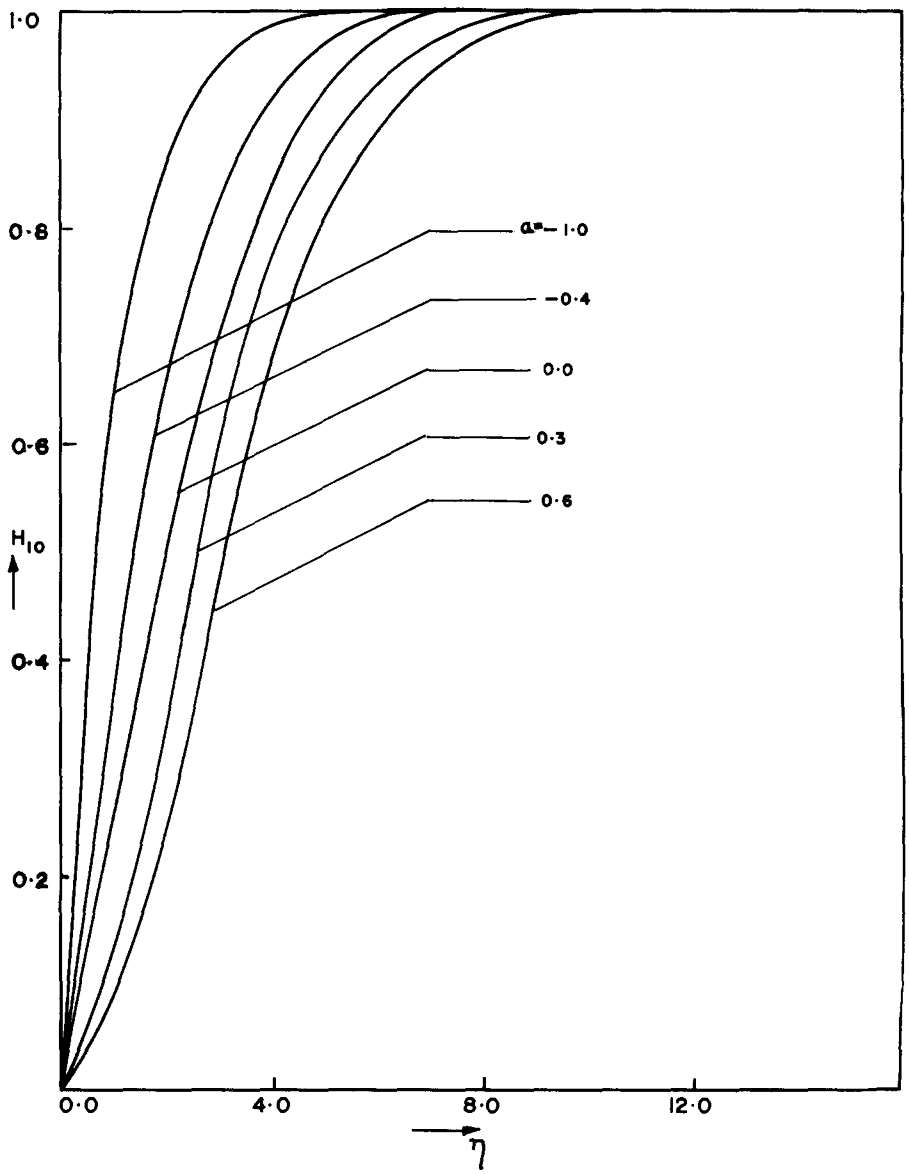

Figure 1. Variation of H_{10} with η for various values of a as indicated.

$H'_{10}(0)$ is almost equal to the magnitude of the suction parameter when a is large. The magnitude of the resultant shear force due to the secondary flow on a rotating disk of radius R is given by

$$\mathcal{F} = \pi R^2 (\rho\mu\Omega)^{1/2} U_0 F. \tag{3.10}$$

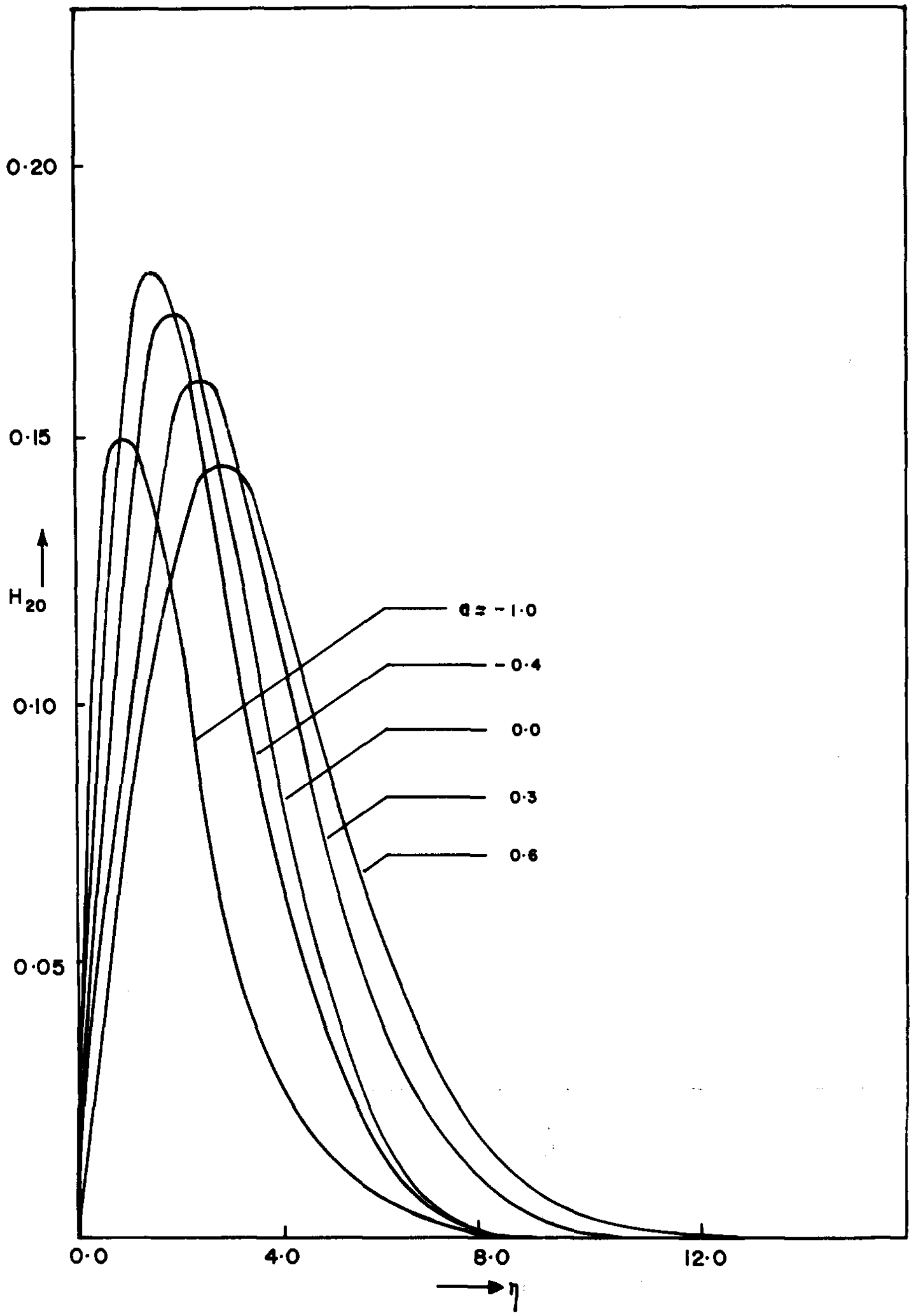

Figure 2. Variation of H_{20} with η for various values of a as indicated.

The force acts at an angle

$$\theta = \tan^{-1}\left(\frac{H'_{20}(0)}{H'_{10}(0)}\right), \tag{3.11}$$

Table 2.

a	F	θ	D
− 4.0	3.99885	− 1.98225	1.99694
− 3.0	2.99762	− 3.40521	1.99046
− 2.5	2.49613	− 4.73158	1.98026
− 2.0	1.99362	− 7.06021	1.95548
− 1.5	1.49119	− 11.37903	1.88758
− 1.2	1.19318	− 15.42160	1.79696
− 1.0	1.00334	− 19.22267	1.70926
− 0.8	0.81642	− 22.64956	1.57610
− 0.4	0.49372	− 30.55616	1.23184
− 0.2	0.36742	− 34.54948	1.04484
0.0	0.26549	− 37.42714	0.86208
0.1	0.20092	− 38.20063	0.69522
0.3	0.15130	− 43.03688	0.60256
0.5	0.10101	− 45.64176	0.46292
0.6	0.0859	− 48.62376	0.39672
0.7	0.06356	− 56.07175	0.33640

with the direction of the free stream. The values of F and θ are given in Table 2. The value of F is almost equal to $|a|$ when $a < -0.8$. The direction of the shearing stress is opposite to that of rotation. When the suction increases the angle made by the secondary shear stress with the direction of the free stream decreases. If T is the torque due to the primary motion, then the ratio of the secondary shear force to the torque is given by

$$\frac{\mathcal{F}}{T} = \frac{2UF}{\Omega R^2 G'(0)} = \frac{U}{\Omega R^2}\ (D). \tag{3.12}$$

The constant $D = 2F/G'(0)$, calculated for different values of a, is given in Table 2. From (3.12) we infer that the secondary flow dominates the primary flow within a radius of $R^* = UD/\Omega$. For specified values of U and Ω, this radius R^* increases if suction is applied and decreases when the fluid is injected through the disk. We also observe that the radius R^* of the 'eye' approaches the value $2U/\Omega$ for large suction. But the region of dominance of the secondary flow shrinks as more and more fluid is injected. Accordingly the secondary-flow calculations for blowing velocities larger than $0.7\sqrt{\nu\Omega}$ are not performed.

When the free stream is oscillating, we have $\beta \neq 0$. Then the corrections to the skin-friction components are time dependent. The windwise and crosswise shear due to the oscillatory free stream are given by the expressions

$$\begin{aligned}\tau_{\text{wind}} = {} & (\rho\mu\Omega)^{1/2} U_0 \{[H'_{10}(0) - \beta^2 H'_{12}(0)] \cos\omega t \\ & - \beta H'_{11}(0) \sin\omega t + O(\beta^3)\},\end{aligned} \tag{3.12}$$

$$\begin{aligned}\tau_{\text{cross}} = {} & (\rho\mu\Omega)^{1/2} U_0 \{[H'_{20}(0) - \beta^2 H'_{22}(0)] \cos\omega t \\ & - \beta H'_{21}(0) \sin\omega t + O(\beta^3)\}.\end{aligned} \tag{3.13}$$

The component of the shear in the free-stream direction has a phase lag over the free stream for all values of a, whereas the shear in the crosswise direction shows a phase lag for $a \leqslant -0.8$ and phase lead for $a > -0.8$.

4. Solution for large frequencies

When the frequency of the free-stream oscillations is large ($\beta \gg 1$), a Stokes shear layer of thickness $\sqrt{\nu/\omega}$ exists on the boundary. For large β, the problem of solving the equations (2.7) and (2.8) becomes a singular perturbation problem. We use the method of composite expansions [15] to obtain the functions H_1 and H_2. Accordingly, the functions H_1 and H_2 are chosen in the form

$$H_1 = e^{-\sqrt{i\beta}\,\eta}\, h_1(\eta) + f_1(\eta), \tag{4.1}$$

$$H_2 = e^{-\sqrt{i\beta}\,\eta}\, h_2(\eta) + f_2(\eta). \tag{4.2}$$

The first part of (4.1) and (4.2) takes into consideration of the Stokes layer and represents the contribution to the flow within the Stokes layer. The second part provides for the flow outside the Stokes layer.

Substituting (4.1) and (4.2) in (2.7) and (2.8) and equating the like terms to zero, we arrive at the following system of equations for h_1, h_2, f_1 and f_2:

$$2h_1' - Hh_1 - (i\beta)^{-1/2}\,[h_1'' + \tfrac{1}{2}H'h_1 - Hh_1' + Gh_2] = 0, \tag{4.3}$$

$$2h_2' - Hh_2 - (i\beta)^{-1/2}\,[h_2'' - Hh_2' + \tfrac{1}{2}h_2H' - Gh_1] = 0, \tag{4.4}$$

$$(f_1 - 1) - (i\beta)^{-1}\,[f_1'' - Hf_1' + \tfrac{1}{2}H'f_1 + Gf_2] = 0, \tag{4.5}$$

$$f_2 - (i\beta)^{-1}\,[f_2'' - Hf_2' + \tfrac{1}{2}H'f_2 - Gf_1] = 0. \tag{4.6}$$

The appropriate boundary conditions are

$$h_1(0) + f_1(0) = 0, \qquad h_2(0) + f_2(0) = 0, \tag{4.7}$$

$$f_1(\infty) = 1, \qquad f_2(\infty) = 0. \tag{4.8}$$

For large β, it is appropriate to expand the function in the form

$$(h_1, h_2, f_1, f_2) = \sum_{n=0}^{\infty} (i\beta)^{-n/2}\,(h_{1n}, h_{2n}, f_{1n}, f_{2n}). \tag{4.9}$$

Substitution of (4.9) in (4.5) and (4.6) yields the outer-flow functions in the exact form as

$$f_{10} = 1, \quad f_{11} = 0, \quad f_{12} = \tfrac{1}{2}H', \quad f_{1n} = 0, \qquad n > 2, \tag{4.10}$$

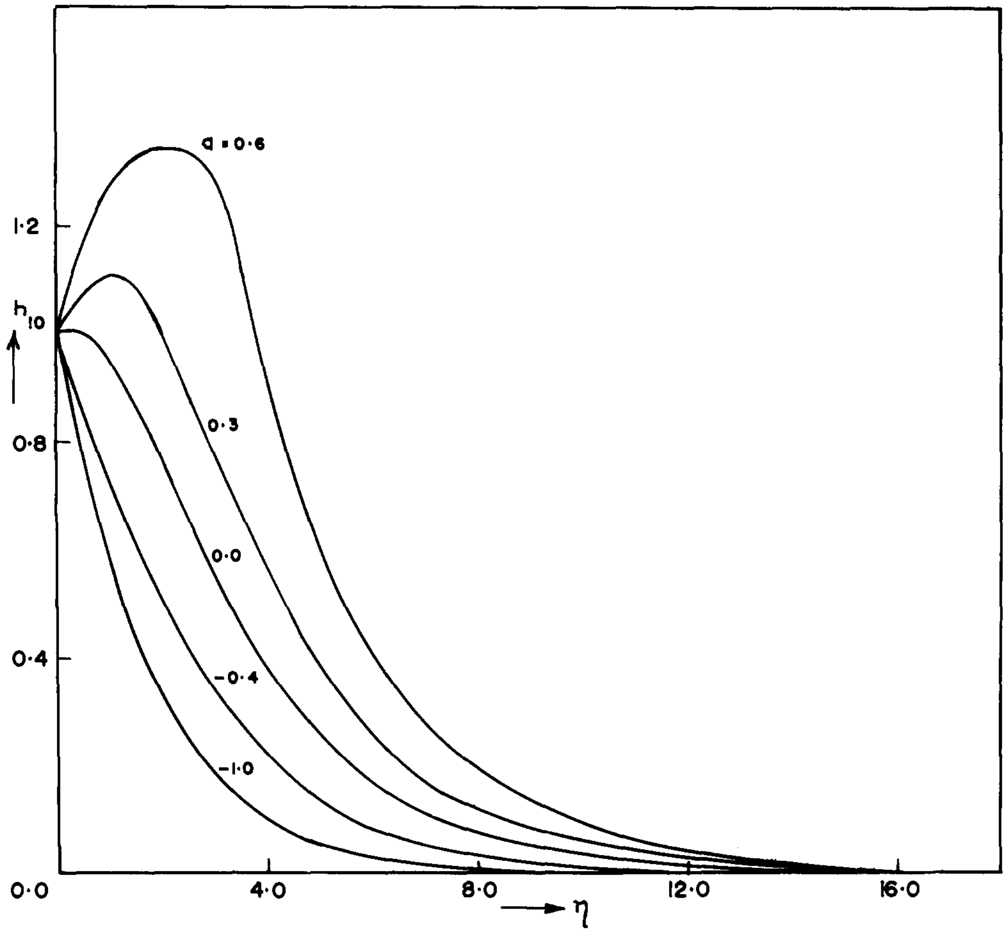

Figure 3. The function h_{10} vs η for values of a shown.

$$f_{20} = 0, \quad f_{21} = 0, \quad f_{22} = -G, \quad f_{2n} = 0, \qquad n > 2. \tag{4.11}$$

The governing equations and the boundary conditions for h_{1n} and h_{2n} are given by

$$2h'_{10} - Hh_{10} = 0, \tag{4.12}$$

$$2h'_{20} - Hh_{20} = 0, \tag{4.13}$$

$$2h'_{1(n+1)} - Hh_{1(n+1)} = h''_{1n} - Hh'_{1n} + \tfrac{1}{2}H'h_{1n} + Gh_{2n}, \qquad n \geqslant 0 \tag{4.14}$$

$$2h'_{2(n+1)} - Hh_{2(n+1)} = h''_{2n} - Hh'_{2n} + \tfrac{1}{2}H'h_{2n} - Gh_{1n}, \qquad n \geqslant 0 \tag{4.15}$$

$$h_{10}(0) = -1, \quad h_{1n}(0) = 0, \qquad n \geqslant 1, \tag{4.16}$$

$$h_{20}(0) = 0, \quad h_{21}(0) = 0, \quad h_{22}(0) = 1, \quad h_{2n}(0) = 0, \qquad n > 2. \tag{4.17}$$

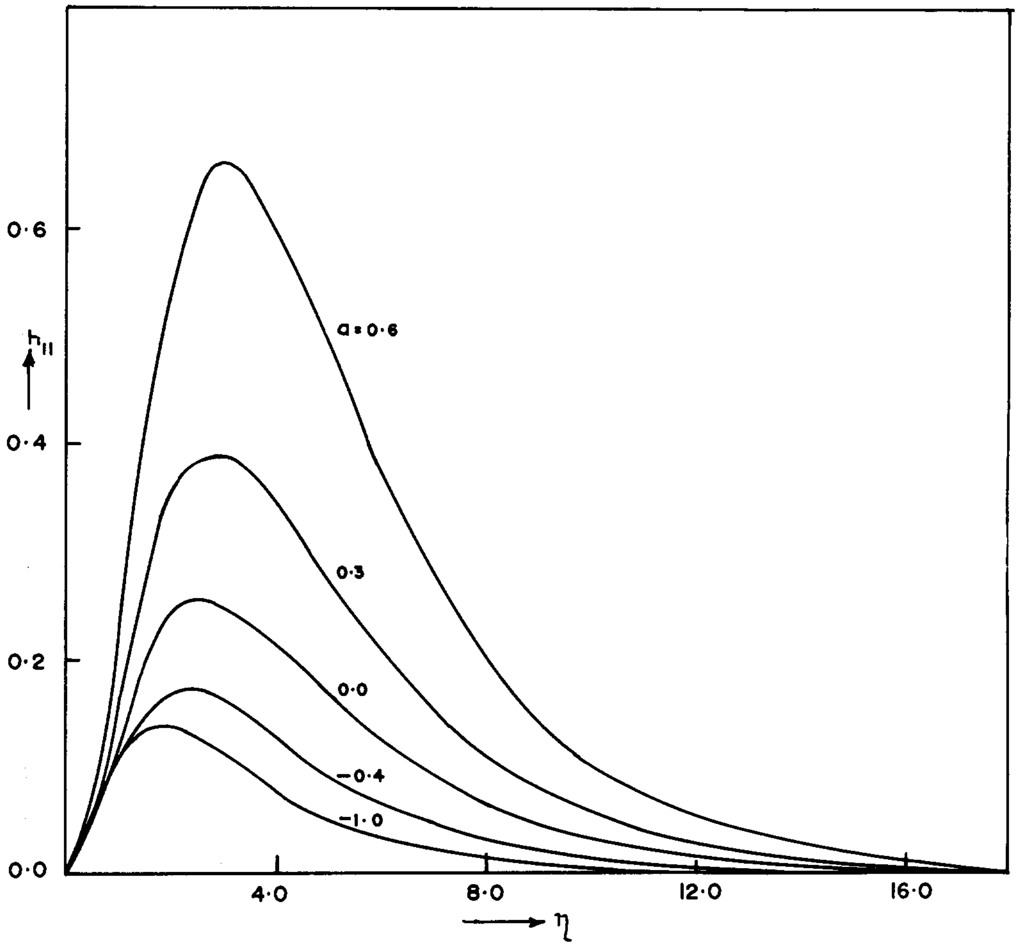

Figure 4. The function h_{11} vs η for values of a shown.

We easily see that $h_{20} = 0$ and the other functions are obtained numerically. The variations of h_{10}, h_{11} and h_{21} with η are shown in Figures 3, 4 and 5 respectively.

We observe that the zeroth-order outer flow due to the pressure gradient in the x-direction produces the flow of $O(\beta^{-1})$ in the streamwise and cross wise directions. Within the Stokes layer a zeroth-order streamwise motion is produced by the corresponding outer flow. This flow, interacting with the primary rotational flow, creates an $O(\beta^{-1/2})$ crosswise flow.

The contribution of the secondary flow to the skin-fraction components can be readily calculated from (4.10), (4.11) and from the equations (4.12) to (4.15). The windwise and the crosswise shear are calculated as

$$\tau_{\text{wind}} = (\rho\mu\Omega)^{1/2} U_0 \left[\beta^{1/2} \cos(\omega t + \pi/4) - \frac{a}{2} \cos \omega t + \frac{a^2}{8} \beta^{-1/2} \cos(\omega t - \pi/4) + \beta^{-1} \frac{H''(0)}{4} \cos(\omega t - \pi/2) + O(\beta^{-2})\right]. \tag{4.19}$$

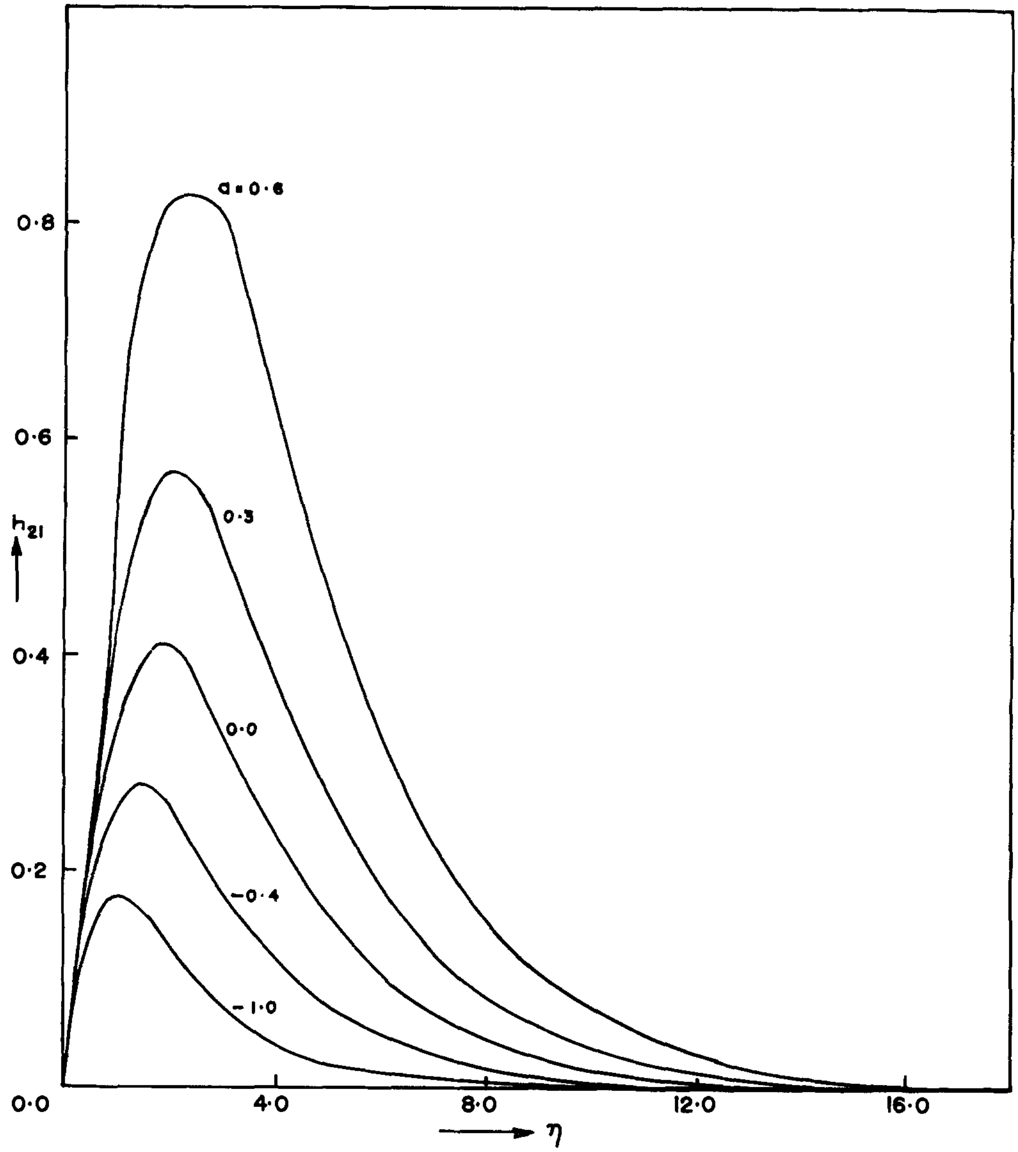

Figure 5. The function h_{21} vs η for values of a shown.

$$\tau_{\text{cross}} = (\rho\mu\Omega)^{1/2} U_0 \left[-\frac{\beta^{-1/2}}{2} \cos(\omega t - \pi/4) + \beta^{-1}\left(\frac{a}{2} - \frac{3}{4} G'(0)\right) \cos(\omega t - \pi/2) + O(\beta^{-2}) \right]. \tag{4.19}$$

The windwise shear is stronger than the crosswise shear. To the leading order, the windwise component of the shear has a phase lead of $\pi/4$ over the free-stream velocity, which is a characteristic of high-frequency flows. As the expressions (4.18) and (4.19) are indicative of a power-series expansion in the positive integral of a, the solution obtained here may not be valid for large values of a. The analysis in that case has to be started with the solution of Stuart [6] or with that of Kuiken [9].

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